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Example 1. (5 Points)
In a radioactive series of 3 different nuclides $n_1(t), n_2(t), n_3(t)$, the rate equations read:

$$\frac{dn_1}{dt} = -\lambda_1 n_1$$
$$\frac{dn_2}{dt} = \lambda_1 n_1 - \lambda_2 n_2$$
$$\frac{dn_3}{dt} = \lambda_2 n_2.$$

a) Write down the **general** Laplace transforms of the differential equations above in terms of the new functions $N_1(p), N_2(p), N_3(p)$. [1.5pt.]
b) Consider the initial conditions $n_1(0) = n_0, n_2(0) = 0, n_3(0) = 0$. Solve the system of equations in Laplace space. [2pt.]
c) Based on the result from b), write down **explicitly** $n_1(t), n_2(t)$ (you do not need to write $n_3(t)$!) [1.5pt.]

Example 2. (5 Points)
We aim at minimizing the following functional:

$$F[y] = \int \frac{\sqrt{dx^2 + dy^2}}{x}.$$

a) Find the function f such that

$$F[y] = \int f(x, y'(x))dx. \text{ [1.5 pt.]}$$

b) Derive the Euler-Lagrange equations for $f(x, y'(x))$, writing the expressions for the derivatives explicitly. [1.5 pt.]
c) Find an expression for $y'(x)$ in terms of x. Keep the constant of integration C! [1pt.]
d) Solve the first-order ODE for $y(x)$ and x using separation of variables. [1pt.]
Hint: Use $Cx = \sin\theta$ to solve the integral, and express the solution in terms of x by noting that $\cos\theta = \sqrt{1 - \sin^2\theta}$.

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Example 3. (5 Points)
For an operator A defined on $\mathcal{D}_A \subset \mathcal{H}$ to be a bounded operator, there must be a finite constant c such that $||Af|| \leq c||f||, \quad \forall f \in \mathcal{D}_A$.
Now let the operator A be defined on $L^2(-\infty, \infty)$:

$$(Af)(x) = h(x)f(x) \quad \text{with} \quad h(x) = \begin{cases} 1, & 1 \leq x \leq 2 \\ 0, & \text{elsewhere.} \end{cases}$$

a) Show that the operator maps as follows $A : L^2(-\infty, \infty) \to L^2(-\infty, \infty)$. [1.5pt]
b) Show that A is linear and bounded operator. What is the constant c? [2pt]
c) Calculate $\mathcal{K}_A$. [1.5pt]

Example 4. (5 Points)
a) Determine the spectrum of the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix}. \text{ [1pt.]}$$

b) Let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by

$$T(x_n)_{n\in\mathbb{N}} = (2x_1, 2x_2 - x_1, 2x_3 - x_2, \dots).$$

Calculate the eigenvectors and eigenvalues of T.[1.5pt.]
c) Determine the adjoint operator of T. [2.5pt.]