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All $\hbar = 1$

42 Points total

Advanced Quantum Mechanics VO
WS 2018/19 Written exam Feb. 5, 2019

Start each exercise on a new sheet of paper. Don't forget to write your name and Matrikelnnumer on each sheet (+2 points if you comply)
You are allowed to use one A4 sheets with your own hand-written formulas (no solved exercises), paper, pen.
Cellphones must be switched off and put in your bag or pocket

1. Scattering within the first Born approximation (8P)

Particles of mass m and momentum $\mathbf{k}$ are scattered by a potential

$$V(\mathbf{r}) = \begin{cases} \beta\, r, \text{ for } r < a \\ 0, \text{ for } r > a \end{cases}$$

Within the first-order Born approximation:
(a) Determine the scattering amplitude $f(\mathbf{k}, \mathbf{k}_{out})$. You don't need to carry out the radial integral.
We consider now the case of small scattering angles θ.
(b) θ must be much smaller than which quantity? (Stating $\theta \ll 1$ is not sufficient).
(c) Determine the differential cross section $d\sigma/d\Omega$ for this case.
(d) Be ρ the particle density. Determine how many particle per unit time are scattered through an angle between $\theta_1 < \theta_2$, both small angles.

2. Two-level boson model (8P)

We consider bosonic particles on a two-level system described by the Hamiltonian

$$\hat{H}_0 = t\left(b_1^\dagger b_2 + b_2^\dagger b_1\right) \qquad t > 0 \tag{1}$$

where $b_i^\dagger/b_i$ are creation/destruction operators for a particle on level i ($i = 1, 2$).
Solve the hamiltonian in the following way:
Introduce two new creation ($d_A^\dagger$, $d_B^\dagger$) and destruction (d_A, d_B) operators related to the b_i by the transformations

$$b_1 = \alpha\,(d_A + d_B) \qquad b_2 = \alpha\,(d_A + \beta\, d_B) \qquad \alpha, \beta = \text{real constants}, \alpha > 0 \tag{2}$$

and their hermitian conjugates.
(a) Given that the d_x ($x = A, B$) obey the correct commutation relations

$$[d_x, d_y^\dagger] = \delta_{x,y}$$

determine for which values of the constants α, β, the b_i, $b_i^\dagger$ obey correct commutation relations as well.
(b) Show that in terms of the d_x the Hamiltonian becomes

$$\hat{H}_0 = p\, d_A^\dagger d_A - q\, d_B^\dagger d_B \qquad p, q > 0 \tag{3}$$

Please turn over

and determine the values of p and q.

3. Eigenstates 8P=1+1+1+1+2+2

(If you haven't solved Ex.2., just use (1),(3),(2) without determining the constants)
In second quantisation write down the following normalized eigenstates of H_0 together with their energies and degeneracies:
(a) the ground state $|G_2\rangle$ of $\hat{H}_0$ with 2 particles,
(b) the first excited state $|E_2\rangle$ with 2 particles.
(c) the ground state $|G_N\rangle$ with N particles,
(d) the first excited state $|E_N\rangle$ with N particles.
Given a perturbation (in second quantisation)

$$\hat{V} = \lambda\, b_1^\dagger b_1 \qquad \lambda = \text{constant}$$

(e) determine the first-order correction to the energy of $|G_N\rangle$.
(f) determine the second-order correction to the energy of $|G_N\rangle$.

4. Two-level fermionic system (6P)

Consider two fermionic particles (with spin) in a system described by two single particle levels $|\varphi_A\rangle, |\varphi_B\rangle$ with energies $\varepsilon_A < \varepsilon_B$.
Write down the following normalized and properly antsymmetrized eigenstates in first quantisation. Express the states as a tensor product $|\text{Orb}\rangle \otimes |S, S_z\rangle$ of an orbital and a spin part, similarly to the treatment of Helium in class. Determine their energy and degeneracy:
(a) The ground state $|G\rangle$
(b) The first excited state(s) $|E\rangle$.
We now introduce creation (and corresponding destruction) operators $c^\dagger_{A,\sigma}, c^\dagger_{B,\sigma}$ creating particle on levels $|\varphi_A\rangle, |\varphi_B\rangle$, respectively.
(c) Write down the states of (a) and (b) in second quantisation (no normalisation necessary here).

5. Field Quantisation (6P)

Given a field $u(x)$ and a Lagrangian

$$L = \int \frac{1}{2}\left(A\,\dot{u}(x)^2 - B\left(\frac{\partial u(x)}{\partial x}\right)^2\right)\,dx$$

(a) Determine the associated momentum field (momentum density) $\pi(x)$.
(b) What commutation rules must be satisfied when quantizing the field u, what is the commutator $[u(x), \dot{u}(y)]$?
(c) Determine the Hamilton operator (in the form $\hat{H} = \int \hat{h}(x)\,dx$)

6. Electromagnetic field (you can use $\hbar = c = q = 1$ here) (6P)

(a) Write the Hamiltonian $\hat{H}$ for a particle of charge q in a homogeneous, time-independent magnetic field B in the z-direction.
(b) Write down the three components of the velocity operator $(\hat{v}_x, \hat{v}_y, \hat{v}_z)$.
(c) Repeat (a) and (b) in a different gauge and write the relation between the wave functions ψ_1, ψ_2 of the two gauges (i.e. $\psi_1 = \text{something} \times \psi_2$.)